

Physically-Relativized Church-Turing Hypotheses

Martin Ziegler*

University of Paderborn, GERMANY

Abstract. We turn ‘the’ Church-Turing Hypothesis from an ambiguous source of sensational speculations into a (collection of) sound and well-defined scientific problem(s):

Examining recent controversies, and causes for misunderstanding, concerning the state of the Church-Turing Hypothesis (CTH), suggests to study the CTH relative to an arbitrary but specific physical theory—rather than vaguely referring to “nature” in general. To this end we combine (and compare) physical structuralism with (models of computation in) complexity theory. The benefit of this formal framework is illustrated by reporting on some previous, and giving one new, example result(s) of computability and complexity in computational physics.

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1 Introduction

In 1937 Alan Turing proposed, and thoroughly investigated the capabilities and fundamental limitations of, a mathematical abstraction and idealization of a computer. This Turing machine (TM) is nowadays considered the most appropriate model of actual digital computers, reflecting what a common PC (say) can do or cannot, and capturing its fundamental in-/capabilities in computability and complexity classes: any computation problem that can in practice be solved (efficiently) on a PC belongs to Δ_1 (to \mathcal{P}); and vice versa. In this sense, the TM is widely believed to be universal; and problems $P \notin \mathcal{P}$, or the Halting problem $H \notin \Delta_1$, have to be faced up to as principally unsolvable in reality.

1.1 Turing Universality in Computer Science and Mathematics

Indeed there is strong evidence for this belief:

- There exists a so-called *universal* Turing machine (UTM), capable of simulating (with at most polynomial slowdown) any other given TM.
- Several other natural, yet seemingly unrelated models of computation have turned out as equivalent to the TM: WHILE-programs, λ -calculus etc. Notice that these correspond to real-world programming languages like `Lisp`!

We qualify those evidence as *computer scientific*—in contrast to the following *mathematical* evidence:

- An integer function f is TM-computable iff it is μ -recursive; that is, f belongs to the least class of functions
 - containing the constant function 0,
 - the successor function $x \mapsto x + 1$,
 - the projections $(x_1, \dots, x_n) \mapsto x_i$,
 - and being closed under composition,
 - under primitive recursion,
 - and under so-called μ -recursion.

Observe that this is a purely (and natural, inner-) mathematical notion indeed.

1.2 Turing Universality in Physics

The* Church-Turing Hypothesis (CTH) claims *that every function which would naturally be regarded as computable is computable under his [i.e. Turing’s] definition, i.e. by one of his machines* [Klee52, p.376]. Its *strong* version claims that *efficient* natural computability corresponds to *polynomial-time* Turing computability. Put differently, CTH predicts a negative answer to the following

Question 1. Does nature admit the existence of a system whose computational power strictly exceeds that of a TM?

Notice that the CTH transcends computer science; in fact, it involves *physics* as the general analysis of nature. Hence, if the answer to Question 1 turned out to be negative, this would establish a third in addition to the above two, computer scientific and mathematical, dimensions of Turing universality (cmp. [Benn95, Svoz05]):

- The class of (efficiently) physically computable functions coincides with the class of (polynomial-time) Turing computable ones.

Indeed, a TM can be built, at least in principle**, and hence constitutes a physical system; whereas a negative answer to Question 1 means that, conversely, *every* physical ‘computer’ can be simulated (maybe even in polynomial time) by a TM. Such an answer is supported by long experience in two ways:

* To be honest, this is just *one* out of a large variety of interpretations of this hypothesis; see, e.g. [Ord02, SECTION 2.2], [Cope02], or [LoCo08]

** it is for instance realized (in good approximation) by any standard PC

- the constant failure to physically solve the Halting problem and
- the success of simulating a plethora of physical systems on a TM, namely in **Computational Physics**.

However so far all attempts have failed to prove the CTH, i.e. have given at best bounds on the *speed* of calculations but not on the general capabilities of computation, based e.g. on the laws of thermodynamics [BeLa85,Fran02] or the speed of light (special relativity) [Lloy02]. In fact it has been suggested that the Church-Turing Hypothesis be included into physics as an *axiom*: just like the impossibility of perpetual motion as a source of energy first started as a recurring experience and was then postulated as the Second Law of Thermodynamics. Either way, whether axiomatizing or trying to prove the Church-Turing Thesis, one first needs a formalization of Question 1.

1.3 Summary

The CTH is the subject of a plethora of publications and of many hot disputes and speculations. The present work aims to put some reason into the ongoing, and often sensational [Kie03b,Lloy06], discussion. We are convinced that this requires formalizing Question 1. However it seems unlikely to reach consensus about one single formalization. In fact we notice that most, if not all, disputes about the state of the Church-Turing Hypothesis arise from disagreeing, and usually only implicit, conceptions of how to formalize it. So what I propose is a *class* of formalizations, namely one for each physical theory.

Manifesto 2. a) *Describing the scientific laws of nature is the purpose and virtue of physics. It does so by means of various physical theories Φ , each of which ‘covers’ some part of reality (but becomes unrealistic on another part).*
 b) *Consequently, instead of vaguely referring to ‘nature’, any claim concerning (the state of) the CTH should explicitly mention the specific physical theory Φ it considers;*
 c) *and criticism against such a claim as ‘based on unrealistic presumptions’ should be regarded as directed towards the underlying physical theory (and stipulate re-investigation subject to another Φ , rather than dismissing the claim itself).*
 d) *Also the input/output encoding better be specified explicitly when referring to some “ CTH^Φ ”: How is the argument \vec{x} , of natural or real numbers, fed into the system; i.e. how does its **preparation** (e.g. in Quantum Mechanics) proceed operationally; and how is the ‘result’ to be read off (e.g. what ‘question’ is the system to answer)? [Ship93, SECTION I]*

The central Item b) explains for the title of the present work; the suggestion to consider physically-relativized Church-Turing Hypotheses “ CTH^Φ ” bears the spirit of the related treatment of the famous “ $\mathcal{P} = \mathcal{N}(\mathcal{P})$ ” Question in [BGS75].

Section 3 below expands on the concept (and notion within the philosophy of science) of a physical theory Φ and its analogy to a model of computation in computer science. We turn Manifesto 2b) into a research programme (Section 5.1) and illustrate its benefit to computational physics. Before, Section 2 reports on previous attempts to disprove the CTH by examples of hypercomputers purportedly capable of solving the Halting problem, and the respective physical theories they exploit. We then significantly simplify one such example to carefully inspect its source of computational power and, based on this insight, are in Section 4.2 led to extend the above

Manifesto 2 (continued).

- e) *The term “exist” in Question 1 must be interpreted in the sense of constructivism.*

2 Physical Computing

Common (necessarily informal) arguments in favor of the Church-Turing Hypothesis usually proceed along the following line: A physical system is mathematically described by an ordinary or partial differential equation; this can be solved numerically using time-stepping—as long as the solution remains regular: whereas a singular solution is unphysical anyway and/or too unstable to be harnessed for physical computing.

On the other hand, the literature knows a variety of suggestions for physical systems of computational power exceeding that of a TM; for instance:

Example 3 i) *General Relativity* might admit for space-times such that the clock of a TM M following one world-line seems to reach infinity within finite time according to the clock of an observer O starting at the same event but following another world-line; O thus can decide whether M terminates or not [EtNe02].

However it is not known whether such space-times actually exist in our universe; and if they do, how to locate them and how far off from earth they might be in order to be used for solving the Halting problem. (Notice that the closest known Black Hole, namely next to star V 4641, takes at least 1600 years to travel to). Finally it has been criticized that, in this approach, a TM would have to actually run indefinitely—and use corresponding amounts of storage tape and energy.

ii) While ‘standard’ quantum computers using a finite number of qubits can be simulated on a TM (although possibly at exponential slowdown), Quantum Mechanics (QM) supports operators on infinite superpositions which may be exploited to solve the Halting problem [CDS00,Kie03a,ACP04,Zieg03]. On the other hand, already finite*** quantum parallelism is in considerable doubt of practicality due to issues of decoherence, i.e. susceptibility to external, classical noise (a kind of instability if you like); hence how much more unrealistic be infinite one!

iii) Certain theories of Quantum Gravitation involve, already in their mathematical formulation, combinatorial conditions which are known undecidable to a TM [GeHa86].

These, however, are still mere (and preliminary) theories...

iv) A light ray passing through a finite system of mirrors corresponds to the computation of a Turing machine; and by detecting whether it finally arrives at a certain position, one can solve the Halting problem [RTY94].

The catch is that the ray must adhere to Geometric Optics, i.e. have infinitely small diameter, be devoid of dispersion, and propagate instantaneously; also the mirrors have to be perfect.

v) The above claim that singular solutions can be ruled out is put into question by the discovery of non-collision singularities in Newtonian many-body systems [Yao03,Smi06a].

On the other hand, the construction of these singularities heavily relies on the moving particles being ideal points obeying Newton’s Law (with the singularity at 0) up to arbitrary small distances.

vi) Even Classical Mechanics has been suggested to allow for physical objects which can be probed in finite time to answer queries “ $n \in X$ ” for any fixed set $X \subseteq \mathbb{N}$ (and in particular for the Halting problem) [BeTu04].

Notice that each approach is based on, and in fact exploits sometimes beyond recognition, some (more or less specific) physical theory. Also, the indicated reproaches against each approach to hypercomputation in fact aim at, and thus challenge the correctness of, the physical theory it is based on.

3 Physical Theories

have been devised for thousands of years as the scientific means for objectively describing, and predicting the behavior of, nature. We nowadays may feel inclined to patronize e.g. ARISTOTLE’s eight books, but his concept of *Elements* (air, fire, earth, water) constitutes an important first step towards putting some structure into the many phenomena experienced†

Since Aristotle, a plethora of physical theories of space-time has evolved (cf. e.g. [Duhe85]), associated with famous names like GALILEO GALILEI, PTOLEMY, NICOLAUS COPERNICUS JOHANNES KEPLER, SIR ISAAC NEWTON, HENDRIK LORENTZ, and ALBERT EINSTEIN. Moreover theories of electricity and magnetism have sprung and later became unified (JAMES CLERK MAXWELL) with GAUSSIAN Optics.

*** The present world-record seems to provide calculations on only 28 qubits; and even that is rather questionable [Pont07]

† Even more, closer observation reveals that an argument like “A rock flung up will fall down, because it is a rock’s nature to rest on earth.” is no less circular than the following two more contemporary ones: “A rock flung up will fall down, because there is a force pulling it towards the earth.” and *Electrons in an atom occupy different orbits, because they are Fermions.*

And there are various[‡] quantum mechanical and field theories. Then the unification process continued: Electricity and Magnetism, been merged into Electrodynamics, were joined by Quantum Mechanics to make up Quantum electrodynamics (QED), and then with Weak Interaction formed Electroweak Interaction; moreover Gravitation and Special Relativity became General Relativity.

Remark 4 (Analogy between a Physical Theory and a Model of Computation). *Each such theory has arisen, or rather been devised, in order to describe with sufficient accuracy some part of nature—while necessarily neglecting others. (Quantum Mechanics for instance is aimed at describing elementary particles moving considerably slower than light; whereas Relativity Theory focuses on very fast yet macroscopic objects.) We point out the analogy of a physical theory to a model of computation in computer science: Here, too, the goal is to reflect some aspects of actual computing devices while being unrealistic with respect to others. (A Turing machine has unbounded working tape and hence can decide whether a 4GB-memory bounded PC algorithm terminates; whereas the canonical model for computing devices with finite memory, a DFA is unable to decide the correct placement of brackets.)*

But what exactly is a physical theory? Agreement on this issue is, in addition to a means for clearing up misunderstandings as indicated in Footnote[‡], a crucial prerequisite for treating important further questions like:

Are Newton's Laws an extension of Kepler's? [Duhe54] Does Quantum Mechanics imply Classical Mechanics—and if so, in what sense exactly?

To us, such intertheory relations [Batt07,Stoe95] are in turn relevant in view of the above Manifesto 2 with questions as the following one:

Do the computational capabilities of Quantum Mechanics include those of Classical Mechanics?

3.1 Structuralism in Physics

Just like a physical theory is regularly obtained by trying to infer a simple description of a family of empirical data points obtained from experimental measurements, a *meta*-theory of physics takes the variety of existing physical theories as empirical data points and tries to identify their common underlying structure. Indeed the philosophy of science knows several meta-theories of physics, that is, conceptions of what a physical theory is [Schm08]:

- SNEED focuses on their mathematical aspects [Snee71]; STEGMÜLLER suggests to formalize physical theories in analogy to the Bourbaki Programme in mathematics [Steg79,Steg86].
- C.F. v. WEIZSÄCKER envisions the success of unifying previously distinct theories (recall above) to continue and ultimately lead into a “Theory of Everything” [Weiz85,Sche97]. To this perspective, any other physical theory (like e.g. Newton Mechanics) is merely a tentative draft [Wein94].
- MITTELSTAEDT emphasizes plurism in physical theories, that is, various theories equally appropriate to describe the same range of phenomena [Mitt72, SECTION 4]. Also [Hage82] points out (among many other things) that any physical theory, or *model*, is a mere approximation and idealization of reality.
- LUDWIG [Ludw90] and, building thereon, SCHRÖTER [Schr96] propose the, for our purpose, most appropriate and elaborated formalization, based on the following (meta-)

Definition 5 (Sketch). *A physical theory Φ consists of*

- *a description of a part of nature it applies to (WB)*
- *a mathematical theory as the language to describe it (MT)*
- *and mapping between physical and mathematical objects (AP).*

[‡] Remember how scientists regularly get into a fight when starting to talk about (their conception of) Quantum Mechanics

Note that, in this setting, each physical theory has a specific and limited range of applicability (WB): a quite pragmatic approach, compared to the almost eschatological conception of von Weizsäcker and Weinberg. The only hope implicit in Definition 5, on the other hand, is that the variety of physical theories keeps augmenting such as their WBs (=images of MTs under APs) eventually ‘cover’ and describe whole nature: just like a mathematical manifold being covered and described by the images of Euclidean subsets under charts [Miln97].

3.2 On the Reality of Physical Theories

The purpose of a physical theory Φ is to describe some part of nature. Hence, if and when some better description Φ' is found, a ‘revolution’ occurs and Φ gets disposed of [Kuhn62]. However this it seems to have happened *lege artis* only very rarely (and is one source of criticism against Kuhn): more commonly, the new theory Φ' is applied to those parts of nature which the old one would not describe (sufficiently well) while keeping Φ for applications where it long has turned as appropriate.

Example 6 a) *Classical/Continuum Mechanics (CM) for instance is often heard of as ‘wrong’—because matter is in fact composed from atoms circled by electrons on stable orbits—yet it still constitutes the theory which most mechanical engineering is based on.*
 b) *Similarly, audio systems are successfully designed using Ohm’s Law for (complex) electrical resistance: in spite of Maxwell’s Equations being a more accurate description of alternating currents, not to mention QED.*

In fact, QM (which the reader might feel tempted to suggest as ‘better’, in the sense of more realistic, a theory than CM) has been proven to *not* include or imply CM [Ludw85]—although such claims regularly re-emerge particularly in popular science. Moreover, even QM itself is again merely[§] an approximation to parts of nature, unrealistic e.g. at high velocities or in the presence of large masses.

These observations urge us to enhance Manifesto 2a+c):

Manifesto 7. *A physical theory Φ (like, e.g. CM) constitutes an ontological entity of its own: It exists no less than “points” or “atoms” do. In particular, it is advisable to investigate the computational power of, and within, such a Φ (and not dismiss it on the grounds of being unrealistic: a tautological feature of any theory). Again we stress the analogy to theoretical computer science (Remark 4) studying the computational power of models of computation M (e.g. finite automata, nondeterministic pushdown automata, linear-bounded nondeterministic Turing machines: the famous **Chomsky Hierarchy** of formal languages) although each such M is unrealistic in some respect.*

4 Hypercomputation in Classical Mechanics?

Let us exemplify Example 3vi) with an alternative ‘hypercomputer’ similar to the one presented in [BeTu04] yet stripped down to purely exhibit, and make accessible for further study, the core idea.

Example 8 *Consider a solid body, a cuboid into which has been carved a ‘comb’ with infinitely many teeth of decreasing width and distance, cf. Figure 1. Moreover, having broken off tooth no. n iff $n \notin H$, we arrive at an encoding of the Halting problem into a physical object in CM.*

This very object (together with some simple mechanical control) is a hypercomputer! Indeed it may be read off, and used to decide for each $n \in \mathbb{N}$ the question “ $n \in H$?” by probing with a wedge the presence of the corresponding tooth.

A first reproach against Example 8 might object that the described system, although capable of solving the Halting problem, is no *hypercomputer*: because it cannot do anything else, e.g. simulate other Turing machines. But this is easy to mend: just attach the system to a universal TM, realized in CM [FrTo82].

[§] In particular we disagree with the, seemingly prevalent, opinion that Quantum Theory is somehow salient or even universal in some sense [HaHa83,Holl96]

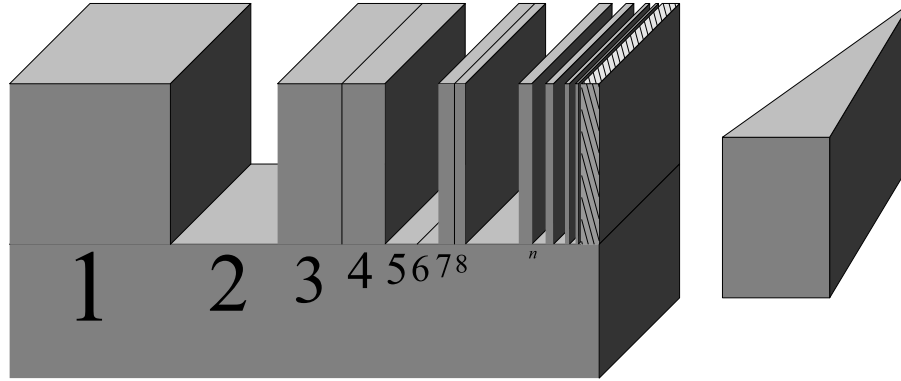


Fig. 1. Infinite comb with a wedge to probe its teeth

The second deficiency of Example 8 is more serious: the concept of a solid body in CM is merely an idealization of actual matter composed from a very large but still finite number of atoms—bad news for an infinite comb. However, as pointed out in Section 3.2, we are to take for serious, and study the computational power of and within, CM as a physical theory.

But even then, there remains an important

Observation 9 (Third issue about Example 8) *Even within CM, i.e. granting the existence of ideal solids and infinite combs, how are we to get hold of one encoding H ? Obviously one cannot construct it from a blank without solving the Halting problem in the first place. Hence our only chance is to simply find one (e.g. left behind by some aliens [Clar68, StSt71]) without knowing how to create one ourselves.*

4.1 Existence in Physics

In order to formalize the Church-Turing Hypothesis (a prerequisite for attempting to settle it), we thus cannot help but notice an ambiguity about the word “*exist*” in Question 1 pointed out already in [Zieg03, REMARK 1.4]: For a physical object to exist within a physical theory, does that mean that

- A) one has to actually *construct* it?
- B) its *non*-existence leads to a contradiction?
- C) or that its *existence* does *not* lead to a contradiction (i.e. is consistent)?

These three opinions correspond in mathematics to the points of view taken by a *constructivist*, a *classical* mathematician (working e.g. in the Zermelo-Fraenkel framework), and one ‘believing’ in the Axiom of Choice, respectively. And at least the last standpoint (C) is well known to lead to counter-intuitive consequences when taken in the physical realm of CM:

Example 10 (Banach-Tarski Paradoxon) *For a solid ball (say of gold) of unit size in 3-space, there exists a partition into finitely many (although necessarily not Lebesgue-measurable) pieces that, when put together appropriately (i.e. after applying certain Euclidean isometries), then form two solid balls of unit size.*

Note that this example is in no danger of causing inflation: on the one hand, because actual material gold is not infinitely divisible (cmp. the second deficiency of Example 8); but even *within* CM, because the partition of the ball ‘exists’ merely in the above Sense C).

Hence, in order to avoid both ‘obviously’ unnatural (counter-) Examples 8 and 10 while sticking to Manifesto 7, we are led to transfer and adapt the constructivist standpoint from mathematics to and for physics.

4.2 Constructivism into Physical Theories!

As explained above, the “existence” of some physical object within a theory Φ is to be interpreted constructively. Let us, similar to [CDCG95], distinguish two ways of introducing constructivism into a physical theory $\Phi = (\text{MT}, \text{AP}, \text{WB})$:

- α) By interpreting the mathematical theory **MT** constructively; compare [BiBr85, Kush84, BrSv00, Flet02] and [DSKS95, SECTION III].
- β) By imposing constructivism onto the side of physical objects **WB**.

It seems that Method α), although meritable of its own, does not quite meet our goal of making a *physical* theory constructive:

Example 11 Consider the condition for a function $f : X \rightarrow Y$ between normed spaces to be open; or even simpler: that of the image $f[B(0, 1)] \subseteq Y$ of the unit ball in X to be an open subset of Y .

$$\forall u \in f[B(0, 1)] \exists \epsilon > 0 \forall y \in B(u, \epsilon) \exists x \in B(0, 1) : f(x) = y . \quad (1)$$

A constructivist would insist that both existential quantifiers be interpreted constructively; whereas in a setting of computation on real numbers by rational approximation, applications suffice that only ϵ be computable from u , while the existence of x depending on y need not: compare [Zieg06].

4.2.1 Constructing Physical Objects

The conception underlying β) is that every object in nature (or more precisely: that part of nature described by **WB**) is

- either a primitive one (e.g. a tree, modeled in Φ as a homogeneous cylinder of density $\rho = 0.7\text{g/cm}^3$; or, say, some ore, modeled as Cu Fe S_2)
- or the result of some technological process applied to such primitive objects.

The latter may for instance include crafting a tree into a wheel or even a wooden gear; or smelting ore to produce bronze.

Notice also how such a process—the sequence of operations from cutting the tree, cleaning, sawing, carving; or of melting, reducing, and alloying copper—constitutes an **algorithm** (and crucial cultural knowledge passed on from carpenters or redsmiths to their apprentices). More modern and advanced science, too, knows (and teaches students) ‘algorithms’ for constructing physical objects: e.g. in mechanical engineering (designing a gear, say) or in QM (using a furnace with boiling silver and some magnets to create a beam of spin- $\frac{1}{2}$ particles as in the famous STERN and GERLACH Experiment and thus operationally *construct* a physical object corresponding via **AP** to a certain wave function ψ as a mathematical object in **MT**).

We are thus led to extend Definition 5:

Definition 12 (Meta-). The **WB** of a physical theory Φ consists of

- a specific collection of primitive objects (**PrimOb**)
- and all so-called constructible objects,

i.e., that can be obtained from primitive ones by a sequence (o_i) of preparatory operations.

- The latter are elements o from a specified collection **PrepOp**.
- Moreover, the sequence (o_i) must be “computable”.

The first two items of Definition 12 are analogous to a mathematical theory **MT** consisting of *axioms* (i.e. claims which are true by definition) and *theorems*: claims which follow from the axioms by a sequence of *arguments*. The last requirement in Definition 12 is to prevent the body in Example 8 from being “constructed” by repeated[†] “breaking off a tooth” as preparatory operations. On the other hand, we seem to be

[†] Like me first, the reader may be tempted to admit only finite sequences of preparatory operations. However this would exclude woodturning a handrail out of a wooden cylinder by letting the carving knife follow a curve, i.e. a continuous sequence

heading for a circular notion: trying to formally capture the *computational* contents of a physical theory Φ required to restrict to ‘constructible’ objects, which in turn are defined as the result of a *computable* sequence of preparatory operations. That circle is avoided as follows

Definition 12 (continued). “*Computability*” here means relative to a pre-theory ϕ to, and to be specified with, Φ .

4.2.2 Pre-Theories: Ancestry among Physical Theories

Recall the above example from metallurgy of redoxing an ore: this may be described by the phlogiston theory (an early form of theoretical chemistry, basically extending Aristotle’s concept of four Elements by a fifth resembling what nowadays would be considered oxygen). Such a ‘chemical’ theory ϕ of its own is required to formulate (yet does not imply) metallurgy Φ , and in particular the algorithm therein that yields to bronze: ϕ is a pre-theory to Φ .

We give some further, and more advanced, examples of pre-theories:

- Example 13.* a) The classical Hall Effect relies on Ohm’s law of electrical direct current as well as on Lorentz’ force law.
- b) The Stern-Gerlach experiment, and the quantum theory of spin Φ it spurred, is based on
- a classical, mechanical theory of a spinning top and precession;
 - some basic theory of (inhomogeneous) magnetism and in particular of Lorentz force onto a dipole
 - an atomic theory of matter (to explain e.g. the particle beam)
 - and even a theory of vacuum (TORRICELLI, VON GUERICKE).
- c) In fact, any quantum theory of microsystems requires [Ludw85] some macroscopic pre-theory in order to describe the devices (furnaces, scintillators, amplifiers, counters) for preparing and measuring the microscopic ensembles under consideration.
- d) BARDEEN, COOPER, and SCHRIEFFER’s Nobel prize-winning BCS-Theory of superconductivity is essentially based on QM
- e) whereas superconducting magnets, in turn, are essential to many particle accelerators used for exploring elementary particles.

The reader is referred to [Schr96, DEFINITION 4.0.8] for a more thorough, and formal, account of this concept.

Observation 14 *Technological progress can be thought of as a directed acyclic graph: a node u corresponds to a physical theory Φ ; and may be based on (one or more) predecessor nodes, pre-theories ϕ to Φ . Put differently, physical theories form nets or logical hierarchies; cmp. [Schr96, VERMUTUNG 14.1.2] and [Stoe95].*

5 Applications to Computational Physics

Computer simulations of physical systems have over the last few decades become (in addition to experimental, applied, and theoretical) an important new discipline of physics of its own. It has, however, received only very little support on behalf of Theoretical Computer Science. Specifically, scientists working in this area (typically highly-skilled programmers with an extensive education in, and excellent intuition for, physics) are highly interested in, and generally ask

Question 15. Why is a specific (class of) physical systems hard (in the sense of computing resources like CPU-time) to simulate? Are our algorithms optimal for them, and in what sense? Which are the principal limits of computer simulation?

Answers to such questions for various physical systems Φ (more precisely: theories in the sense of Section 3) are highly appreciated in Computational Physics; answers given of course in the language of, and using methods from, Computational Complexity Theory [Papa94], namely locating Φ in some complexity (or recursion theoretic) class *and* proving it complete for that class.

We observe that, apart from sensational attempts [Lloy06], there are rather few serious and rigorous answers to such questions to-date [FLS05, Wolf85, Moor90, Ship93, Svoz93, ReTa93, RTY94, PIM06, Loff07]. One reason therefor might be that, as opposed to classical problems considered in computational complexity, those arising in Computational Physics naturally involve real numbers [PERi89, WeZh02, WeZh06] where uncomputability easily occurs without completeness [Grze57, Wolf85, Moor90, Smi06a, MeZi06]. On the other hand, there is a well-established theory of bit-complexity and (e.g. $\mathcal{N}P$ -) completeness over \mathbb{R} [Frie84, Ko91]. Moreover for problems *defined* over real numbers but restricted to rational inputs, the situation can become quite subtle (and interesting): see, e.g., [CCK*04] or [Zieg06, PROPOSITION 30].

5.1 Sketch of A Research Programme

We propose a systematic exploration of the computational power (i.e. completeness) of a large variety of physical theories. The first goal is a general picture of physically-relativized Church-Turing Hypotheses, that is, on the boundary between decidability and Turing-completeness; later one may turn to lower complexity classes like EXP , $PSPACE$, NP , P , and NC . The focus be on a thorough investigation, starting from simplest, decidable theories and slowly proceeding towards more complex ones (not necessarily in historical order) rich enough to admit a Turing- (i.e. Δ_2 -) complete system therein. In particular, it seems advisable to begin with rather modest (rather than straight away with sexy ‘new’) physics:

5.2 Celestial Mechanics

Recall the historical progress of describing and predicting the movement of planets and stars observed in sky from Eudoxus/Aristotle via Ptolemy, Copernicus, and Kepler to Newton and Einstein. Indeed, these descriptions constitute (not necessarily comparable, in the sense of reduction) physical theories! The present subsection exemplifies our proposed approach by investigating and reporting on the computational complexities of two of them. (We admit that, lacking any option for preparation, celestial mechanics is of limited use as a computational system in the sense of Manifesto 2d).

5.2.1 Newton

Consider a physical theory Φ of N points moving in Euclidean 3-space under mutual attracting force proportional to distance⁻² (inverse-square law). This is the case for **Electrostatics** (Coulomb) as well as for **Classical Gravitation** (Newton). Some questions, in the sense of Manifesto 2d), may ask:

1. Does point #1 reach within one second the unit ball B centered at the origin?
2. Does some point eventually escape to infinity?
3. Do two points (within 1sec or ever) collide?

It has been argued that Question c) makes not much sense, because a ‘collision’ of *ideal* points (recall Manifesto 7) can be analytically continued to just pass through each other. Note that Question a) is not ‘well-posed’ in case the point just touches the boundary of B ; it is therefore usually accompanied by the *promise* that point #1 either meets the interior of B within one second or avoids the blown-up ball $2B$ for two seconds; and shown $PSPACE$ -hard in this case [ReTa93]. Question b) has only recently been shown to make sense in that a positive answer is actually possible [Xia92]; and it has been shown undecidable [Smi06a]—however for input configurations described by (possibly transcendental) *real* numbers given as *infinite* sequences of rational approximations: for such encodings, mere discontinuity is known to trivially imply uncomputability without completeness [Grze57].

5.2.2 Planar Eudoxus/Aristotle

An early theory of celestial mechanics originates from ancient Greece. An important purpose of it, and also of its successors (see Section 5.2.4 below), was to describe and predict the movement, and in particular **conjunctions**, of planets and stars. Let us capture this, distinguishing short-term from long-term behavior [Ship93, SECTION I], in the following

Question 16. 1. Will certain planets attain perfect conjunction, ever?

2. or within a given time interval?
3. or reach an approximate conjunction, i.e. meet up to some prescribed angular distance ϵ ?

According to ARISTOTLE (Book Λ of *Metaphysics*) and EUDOXUS OF CNIDUS, earth resides in the center of the universe (recall the beginning of Section 3) and is circled by *celestial spheres* moving the celestial bodies.

Definition 17. Let Φ denote the physical theory (which we refrain from fully formalizing in the sense of Definition 5 or even [Schr96]) parameterized by the initial positions u_i of planets $i = 1, \dots, N$, and their constant directions \vec{d}_i and velocities v_i of rotation.

By Φ' , we mean a two-dimensionally restricted version: planets rotate on circles perpendicular to one common direction; compare Figure 2. Moreover, initial positions and angular velocities are presumed ‘commensurable’^{||}, that is, rational (multiples of π).



Fig. 2. Celestial orbs as drawn in PETER APIAN’s *Cosmographia* (Antwerp, 1539)

Recall that $\mathcal{NC} \subseteq \mathcal{P}$ is the class of problems solvable in polylogarithmic parallel time on polynomially many processors; whereas \mathcal{P} -hard problems (w.r.t. *logspace*-reductions, say) presumably do not admit such a beneficial parallelization. The greatest common divisor $\gcd(a, b)$ of two given (say, n -bit) integers can be determined** in polynomial time; it is however not known to belong to \mathcal{NC} nor be \mathcal{P} -hard; the same holds for the calculation of an extended Euclidean representation “ $a \cdot y + b \cdot z = \gcd(a, b)$ ”, i.e. of $(y, z) = \text{gcdex}(a, b)$ [GHR95, B.5.1].

After these preliminaries, we are able to state the computational complexity of the above theory Φ' ; more precisely: the complexity, in terms of Φ' ’s parameters, of the decision problems raised in Question 16:

Theorem 18. Let $k \leq n \in \mathbb{N}$ and $u_1, \dots, u_n, v_1, \dots, v_n \in \mathbb{Q}$ be given initial positions and angular velocities (measured in multiples of 2π) of planets #1, ..., #n in Φ' .

^{||} We don’t want anybody to get drowned like, allegedly, HIPPARCUS OF METAPONTUM. Also, since rational numbers are computable, we thus avoid the issues from Section 4.2.

** The attentive reader will connive our relaxed attitude concerning decision versus function problems

- a) Planets #1 and #2 will eventually appear in perfect conjunction iff $v_1 \neq v_2 \vee u_1 = u_2$.
b) Planets #1 and #2 appear closer than $\varepsilon > 0$ to each other within time interval (a, b) iff it holds, in interval notation:

$$\emptyset \neq \mathbb{Z} \cap ((a, b) \cdot (v_1 - v_2) + u_1 - u_2 + (-\varepsilon, +\varepsilon)) .$$

This can be decided within \mathcal{NC}^1 .

- c) The question of whether all planets #1, ..., #n will ever attain a perfect conjunction, can be decided in $\mathcal{NC}^{\text{gcd}}$;
d) and if so, the next time t for this to happen can be calculated in $\mathcal{NC}^{\text{gcdex}}$.
e) Whether there exist k (among the n) planets that ever attain a perfect conjunction, is \mathcal{NP} -complete a problem.

5.2.3 Proofs

The major ingredient is the following tool concerning the computational complexity of problems about rational arithmetic progressions:

Definition 19. For $u, v \in \mathbb{Q}$, let $u \div v := (a \div b)/q$ and $\text{gcd}(u, v) := \text{gcd}(a, b)/q$ where $a, b, q \in \mathbb{Z}$ are such that $u = a/q$ and $v = b/q$ and $1 = \text{gcd}(a, b, c)$; similarly for $\text{lcm}(u, v)$.

For $a, \alpha \in \mathbb{Q}$, write $P_{a, \alpha} := \{\alpha + a \cdot v : v \in \mathbb{Z}\}$.

- Lemma 20.** a) Given $a, \alpha \in \mathbb{Q}$, the unique $0 \leq \alpha' < a$ with $P_{a, \alpha} = P_{a, \alpha'}$ can be calculated as $\alpha' := \alpha \text{ rem } a$ within complexity class \mathcal{NC}^1 .
b) Given a, α and b, β , the question whether $P_{a, \alpha} \cap P_{b, \beta} = \emptyset$ can be decided in $\mathcal{NC}^{\text{gcd}}$
c) and, if so, c, γ with $P_{a, \alpha} \cap P_{b, \beta} = P_{c, \gamma}$ can be calculated in $\mathcal{NC}^{\text{gcdex}}$.
d) Items b) and c) extend from two to the intersection of k given arithmetic progressions.
e) Given n and $a_1, \alpha_1, \dots, a_n, \alpha_n$, determining the maximum number k of arithmetic progressions $P^{(i_1)} := P_{a_{i_1}, \alpha_{i_1}}, \dots, P^{(i_k)} := P_{a_{i_k}, \alpha_{i_k}}$ that have nonempty common intersection, is \mathcal{NP} -complete.

A result similar to the last item has been obtained in [MaHa94]...

- Proof.* a) Notice that $P_{a, \alpha} = P_{a, \alpha'} \Leftrightarrow \alpha - \alpha' \in P_{a, 0}$. Hence there exists exactly one such α' in $[0, a)$, namely $\alpha' = \alpha \text{ rem } a$. Moreover, integer division belongs to \mathcal{NC} [BCH86, CDL01].
b) Observe that $P_{a, \alpha} \cap P_{b, \beta} \neq \emptyset$ holds iff $\text{gcd}(a, b)$ divides $\alpha - \beta$. Indeed, the extended Euclidean algorithm then yields $z'_1, z'_2 \in \mathbb{Z}$ with $\text{gcd}(a, b) = -a \cdot z'_1 + b \cdot z'_2$; then $\alpha - \beta = -a \cdot z_1 + b \cdot z_2$ yields $P_{a, \alpha} \ni \alpha + a \cdot z_1 = \beta + b \cdot z_2 \in P_{b, \beta}$. Conversely $\alpha + a \cdot z_1 = \beta + b \cdot z_2 \in P_{a, \alpha} \cap P_{b, \beta}$ implies that $\alpha - \beta = -a \cdot z_1 + b \cdot z_2$ is a multiple of any (and in particular the greatest) common divisor of a and b .
c) Notice that $c = \text{lcm}(a, b) = a \cdot b / \text{gcd}(a, b)$; and, according to the proof of b), $\gamma := \alpha + a \cdot z_1$ will do, where $z_1, z_2 \in \mathbb{Z}$ with $\alpha - \beta = -a \cdot z_1 + b \cdot z_2$ result from the extended Euclidean algorithm applied to (a, b) .
d) Notice that

$$x \in P_{a_1, \alpha_1} \cap \dots \cap P_{a_k, \alpha_k} \Leftrightarrow x \equiv \alpha_i \pmod{a_i}, \quad i = 1, \dots, k . \quad (2)$$

According to the Chinese Remainder Theorem, the latter congruence admits such a solution x iff $\text{gcd}(a_i, a_j)$ divides $\alpha_i - \alpha_j$ for all pairs (i, j) .

In order to calculate such an x , notice that a straight-forward iterative $P_{a_{1..k-1}, \alpha_{1..k-1}} \cap P_{a_k, \alpha_k}$ fails as it does not parallelize well, and also the numbers calculated according to c) in may double in length in each of the k steps. Instead, combine the P_{a_i, α_i} in a binary way first two tuples $P_{a_{2j-1}, \alpha_{2j-1}}, P_{a_{2j}, \alpha_{2j}}$ of adjacent ones, then on to quadruples and so on. At logarithmic depth (=parallel time), this yields the desired result $x =: \alpha_0$ and $a_0 := \text{lcm}(a_1, \dots, a_k)$ satisfying $P_{a_0, \alpha_0} = \bigcap_{i=1}^k P_{a_i, \alpha_i}$.

- e) It is easy to guess i_1, \dots, i_k and, based on d), verify in polynomial time that $P^{(i_1)} \cap \dots \cap P^{(i_k)} \neq \emptyset$.
We establish \mathcal{NP} -hardness by reduction from Clique [GaJo79]: Given a graph $G = ([n], E)$, choose $n \cdot (n-1)/2$ pairwise coprime integers $q_{i, \ell} \geq 2$, $1 \leq i < \ell \leq n$; for instance $q_{i, \ell} := p_{i+n \cdot (\ell-1)}$ will do, where p_m denotes the m -th prime number, found in time polynomial in $n \leq |G|$ (though not in $|\langle p_m \rangle| \approx \log m + \log \log m$) by simple exhaustive search. Then calculate $a_i := \prod_{\ell \neq i} q_{i, \ell}$ and observe that

$\gcd(a_i, a_j) = q_{i,j}$ for $i \neq j$. Now start with $\alpha_1 := 0$ and iteratively for $\ell = 2, 3, \dots, n$ determine α_ℓ by solving the following system of simultaneous congruences:

$$\alpha_\ell \equiv \begin{cases} \alpha_i & (\text{mod } q_{\ell,i}) \text{ for } (\ell, i) \in E \\ 1 + \alpha_i & (\text{mod } q_{\ell,i}) \text{ for } (\ell, i) \notin E \end{cases}, \quad 1 \leq i < \ell \quad (3)$$

Indeed, as the $q_{\ell,i}$ are pairwise coprime, the Chinese Remainder Theorem asserts the existence of a solution—computable in time polynomial in n , regarding that α_ℓ can be bounded by $\prod_{i,j} q_{i,j}$ having a polynomial number of bits). The thus constructed vector $(\alpha_i)_i$ satisfies:

$$\alpha_i \equiv \alpha_j \pmod{\underbrace{\gcd(a_i, a_j)}_{=q_{i,j}}} \Leftrightarrow (i, j) \in E$$

because, for $(i, j) \notin E$, Equation (3) implies $\alpha_i \equiv \alpha_j + 1 \pmod{q_{i,j}}$.

We claim that this mapping $G \mapsto (a_i, \alpha_i : 1 \leq i \leq n)$ constitutes the desired reduction: Indeed, according to Equation (2), any sub-collection $P^{(i_1)}, \dots, P^{(i_k)}$ has non-empty intersection (i.e. a common element x) iff $\alpha_{i_\ell} \equiv \alpha_{i_j} \pmod{\gcd(a_{i_\ell}, a_{i_j})}$, i.e., by our construction, iff $(i_\ell, i_j) \in E$; hence cliques of G are in one-to-one correspondence with subcollections of intersecting arithmetic progressions. \square

Proof (Theorem 18). At time t , planet $\#i$ appears at angular position $u_i + t \cdot v_i \pmod{1}$; and an exact conjunction between $\#i$ and $\#j$ occurs whenever $u_i + t \cdot v_i = u_j + t \cdot v_j + z$ for some $z \in \mathbb{Z}$, that is iff

$$t \in \left\{ \frac{u_j - u_i}{v_i - v_j} + z \cdot \frac{1}{v_i - v_j} \right\} = P^{(i,j)} := P_{a_{i,j}, \alpha_{i,j}} \quad \text{where } a_{i,j} := \frac{1}{v_i - v_j}, \alpha_{i,j} := \frac{u_j - u_i}{v_i - v_j}. \quad (4)$$

Therefore, planets $\#1, \dots, \#n$ attain a conjunction at some time t iff $t \in \bigcap_{i=1}^n P^{(1,i)}$. The existence of such t thus amounts to the non-emptiness of the joint intersection of arithmetic progressions and can be decided in the claimed complexity according to Lemma 20b+d). Moreover, Lemma 20a+c+d) shows how to calculate the smallest t .

Concerning $\mathcal{N}P$ -hardness claimed in Item f), we reduce from Lemma 20e): Given n arithmetic progressions $P^{(i)} = P_{a_i, \alpha_i}$, let $u_i := -\alpha_i \cdot a_i$, $v_i := 1/a_i$, and $u_0 := 0 =: v_0$. Then conjunctions between $\#0$ and $\#i$ occur exactly at times $t \in P^{(i)}$; and $P^{(i_1)}, \dots, P^{(i_k)}$ meet iff (and when/where) $\#0, \#i_1, \dots, \#i_k$ do.

Approximate conjunction up to ε in time interval (u, v) means:

$$\exists t \in (u, v) \exists z \in \mathbb{Z} : (v_2 - v_1)t + u_2 - u_1 + z \in (-\varepsilon, +\varepsilon)$$

which is equivalent to Claim b). The boundaries of the interval $(a, b) \cdot (v_1 - v_2) + u_1 - u_2 + (-\varepsilon, +\varepsilon)$ can be calculated in \mathcal{NC}^1 . \square

5.2.4 General Eudoxus/Aristotle; Ptolemy, Copernicus, and Kepler

Proceeding from the restricted 2D theory Φ' to Eudoxus/Aristotle's full Φ obviously complicates the computational complexity of the above predictions; and it seems desirable to make that precise, e.g. with the help of [ACG93, GaOv95, BrKi98]. Moreover also Φ in turn had been refined: PTOLEMY introduced additional so-called *epicycles* and *deferents*, located and rotating on the originally earth-centered spheres. This allowed for a more (parameters to fit in order to yield an) accurate description of the observed planetary motions. Copernicus relocated the spheres (and sub-spheres thereon) to be centered around the sun, rather than earth. And Kepler replaced them with ellipses in space. Again, the respective increase in complexity is worth-while investigating.

5.3 Opticks

There is an abundance of (physical theories giving) explanations for optical phenomena; cmp. e.g. *The Book of Optics* by IBN AL-HAYTHAM (1021) or NEWTON's book providing the title of this section. We are specifically interested in the progression from geometric via Gaussian (taking into account dispersion) over HUYGENS and FOURIER (diffractive, wave) optics to Maxwell's theory of electromagnetism; and even, in

order to describe the various kinds of scattering observed, to quantum and quantum field theories. Note that this sequence of optical theories Φ_i reflects their historical succession, but *not* a logical one in the sense that Φ_{i+1} ‘implies’ (and hence is computationally at least as hard as) Φ_i .

Our purpose is thus to explore more thoroughly the computational complexities of these theories. In fact their computational relations may happen to be similar, unrelated, or just opposite to their historical ones! Consider for example geometric optics versus Electrodynamics:

5.3.1 Geometric Optics considers light rays as ideal geometric objects, i.e., of infinitesimal section proceeding instantaneously and straightly until hitting a, say, mirror. Now depending on the kind of mirrors (straight or curved, with rational or algebraic parameters) and the availability of further optical devices (lenses, beam splitters), [RTY94] has developed a fairly exhaustive taxonomy of the induced computational complexities of ray tracing ranging from \mathcal{PSPACE} to *undecidable*!

5.3.2 Electrodynamics on the other hand treats light as a vector-valued wave obeying a system of linear partial differential equations named after JAMES CLERK MAXWELL. Their solution, from given initial conditions, is computable, even over real numbers [WeZh99]!

5.4 Quantum Mechanics

is, since RICHARD P. FEYNMAN’s famous *Lectures on Computation* [FLS05], of particular interest to the theory of computation and has, in connection with the work of PETER SHOR’s, initiated Quantum Computation as a now fashionable and speculative [Kie03a] research topic lacking a general picture [Smi06b, Myrv95, WeZh06]. Speaking in complexity theoretic terms, the (as usual highly ambiguous) question raised by the strong CTH (recall Section 1.2) asks to locate the computational power of QM somewhere among (or between) \mathcal{P} , $\mathcal{P}^{\text{IntegerFactorization}}$, $\mathcal{N}\mathcal{P}$, and Δ_2 . And it seems worth-while to further explore how the answer depends on the underlying Hamiltonians being un-/bounded as indicated in [PERi89, CHAPTER 3]?

In order for a sound and more definite investigation, our approach suggests to start exploring well-specified sub- and pre-theories of QM. These may for instance be the BOHR-SOMMERFELD theory of classical electron orbits with integral action-angle conditions.

Another promising direction considers computational capabilities of, and complexity in, Quantum Logic:

5.4.1 Quantum Logic arises as an abstraction of the purely algebraic structure exhibited by the collection of *effect* operators introduced by G. LUDWIG on a Hilbert space (i.e. certain quantum mechanical observables); cf. e.g. [Svoz98]. This discipline has flourished from the comparison with (i.e. systematic and thorough investigation of similarities and differences to) Boolean logic. In particular the axioms satisfied by operations “ \wedge, \vee, \neg, \leq ” differ from the classical case, depending on which quantum logic one considers.

It seems interesting to devise a theory of computational complexity similar to that of Boolean circuits [Papa94, SECTIONS 4.3 and 11.4] with classical gates replaced by quantum *logic* ones. A first important and non-trivial result has been obtained in [DHMW05, SECTION 3] and may be interpreted as: the satisfiability problem for quantum logic gates is decidable.

References

- [ACG93] M.J. ATALLAH, P. CALLAHAN, M.T. GOODRICH: “ \mathcal{P} -Complete Geometric Problems”, pp.443–462 in *Int. J. Comput. Geometry Appl.* vol.**3**:4 (1993).
- [ACP04] V.A. ADAMYAN, C.S. CALUDE, B.S. PAVLOV: “Transcending the limits of Turing computability”, pp.119–137 in *Quantum Information Complexity. Proc. Meijo Winter School 2003* (T. Hida, K. Saito, S. Si Editors), World Scientific Singapore (2004).
- [Batt07] R. BATTERMAN: “Intertheory Relations in Physics”, in *Stanford Encyclopedia of Philosophy*, <http://plato.stanford.edu/entries/physics-interrelate> (2007).
- [BCH86] P. BEAME, S. COOK, H. HOOVER: “Log Depth Circuits for Division and Related Problems”, pp.994–1003 in *SIAM Journal of Computing* vol.**15** (1986).

- [BeLa85] C.H. BENNETT, R. LANDAUER: “The Fundamental Physical Limits of Computation”, pp.48–56 in *Scientific American*, vol.**253**(1) (1985).
- [Benn95] C.H. BENNET: “Universal Computation and Physical Dynamics”, pp.268–273 in *Physica D* vol.**86** (1995).
- [BeTu04] E.J. BEGGS, J.V. TUCKER: “Computations via Experiments with Kinematic Systems”, *Technical Report* 5–2004, Department of Computer Science, University of Wales Swansea.
- [BiBr85] E. BISHOP, D.S. BRIDGES: “*Constructive Analysis*”, Springer (1985).
- [BGS75] T. BAKER, J. GILL, R. SOLOVAY: “Relativizations of the $\mathcal{P} = ? \mathcal{NP}$ Question”, pp.431–442 in *SIAM Journal on Computing* vol.**4**:4 (1975).
- [Breu02] T. BREUER: “*Quantenmechanik: Ein Fall für Gödel?*”, Spektrum (2002).
- [BrKi98] H. BREU, D.G. KIRKPATRICK: “Unit Disk Graph Recognition is \mathcal{NP} -hard”, pp.3–24 in *Computational Geometry Theory and Applications* vol.**9** (1998).
- [BrSv00] D. BRIDGES, K. SVOZIL: “Constructive Mathematics and Quantum Physics”, pp.503–515 in *International Journal of Theoretical Physics* vol.**39**:3 (2000).
- [CCK*04] E.-C. CHANG, S.W. CHOI, D.Y. KWON, H.J. PARK, C.K. YAP: “Shortest Path amidst Disc Obstacles is Computable”, pp.567–590 in *Int. J. Comput. Geometry Appl* vol.**16** (2006).
- [CDL01] A. CHIU, G.I. DAVIDA, B.E. LITOW: “Division in logspace-uniform \mathcal{NC}^1 ”, pp.259–275 in *Informatique Théorique et Applications* vol.**35**:3 (2001).
- [CDS00] C.S. CALUDE, M.J. DINNEEN, K. SVOZIL: “Reflections on Quantum Computing”, pp.35–37 in *Complexity* vol.**6**:1 (2000).
- [CDCG95] G. CATTANEO, M.L. DALLA CHIARA, R. GIUNTINI: “Constructivism and Operationalism in the Foundations of Quantum Mechanics”, pp.21–31 in [DSKS95].
- [Clar68] A.C. CLARKE: “*2001: A Space Odyssey*”, (1968).
- [Cope02] J. COPELAND: “Hypercomputation”, pp.461–502 in *Minds and Machines* vol.**12**, Kluwer (2002).
- [Davi01] E.B. DAVIES: “Building Infinite Machines”, pp.671–682 in *Brit. J. Phil. Sci.* vol.**52** (2001).
- [DSKS95] W. DEPAULI-SCHIMANOVICH, E. KÖHLER, F. STADLER (Eds.): “The Foundational Debate: Complexity and Constructivity in Mathematics and Physics”, *Vienna Circle Institute Yearbook*, Kluwer (1995).
- [Duhe85] P. DUHEM: “*Le système du monde. Histoire des doctrines cosmologiques de Platon à Copernic*”, Paris (1913); partial English translation by R. ARIEW: “*Medieval Cosmology: Theories of Infinity, Place, Time, Void, and the Plurality of Worlds*”, University of Chicago Press (1985).
- [Duhe54] P. DUHEM: “*La théorie physique, son objet, sa structure*”, Paris: Cevalier & Rivière (1906); English translation by P.P. WIENER: “*The Aim and Structure of Physical Theory*”, Princeton University Press (1954).
- [DHMW05] J.M. DUNN, T.J. HAGGE, L.S. MOSS, Z. WANG: “Quantum Logic as Motivated by Quantum Computing”, pp.353–359 in *Journal of Symbolic Logic* vol.**70**:2 (2005).
- [EtNe02] G. ETESI, I. NÉMETI: “Non-Turing Computations Via Malament-Hogarth Space-Times”, pp.341–370 in *International Journal of Theoretical Physics* vol.**41**:2 (2002).
- [Flet02] P. FLETCHER: “A Constructivist Perspective on Physics”, pp.26–42 in *Philosophia Mathematica* vol.**10** (2002).
- [FLS05] R.P. FEYNMAN, R.B. LEIGHTON, M. SANDS: “*The Feynman Lectures on Computation including Feynman’s Tips on Physics*” (2nd Edition), Addison Wesley (2005).
- [FLT82] E. FREDKIN, R. LANDAUER, T. TOFFOLI (Edts): *Proc. Conf. Physics of Computation*:
 “Part I: Physics of Computation”, *International Journal of Theoretical Physics* vol.**21** Nos.3+4 (1982)
 “Part II: Computational Models of Physics”, *Int. Journal of Theoretical Physics* vol.**21** Nos.6+7 (1982)
 “Part III: Physical Models of Computation”, *Int. Journal of Theoretical Physics* vol.**21** No.12 (1982).
- [Fran02] M.P. FRANK: “The Physical Limits of Computing”, pp.16–26 in *IEEE Computing in Science and Engineering* vol.**4**:3 (2002).
- [Frie84] H. FRIEDMAN: “The Computational Complexity of Maximization and Integration”, pp.80–98 in *Advances in Mathematics* vol.**53** (1984).
- [FrTo82] E. FREDKIN, T. TOFFOLI: “Conservative Logic”, pp.219–253 in *International Journal of Theoretical Physics* vol.**21** (1982).
- [GaOv95] A. GAJENTAAN, M.H. OVERMARS: “On a Class of $O(n^2)$ Problems in Computational Geometry”, pp.165–185 in *Comput. Geom.* vol.**5** (1995).
- [GaJo79] M.R. GAREY, D.S. JOHNSON: “*Computers and Intractability: A Guide to the Theory of \mathcal{NP} -Completeness*”, Freeman (1979).
- [GeHa86] R. GEROCH, J.B. HARTLE: “Computability and Physical Theories”, pp.533–550 in *Foundations of Physics* vol.**16**(6) (1986).
- [GHR95] R. GREENSLAW, H.J. HOOVER, W.L. RUZZO: “*Limits to Parallel Computation*”, Oxford University Press (1995).

- [Grze57] A. GRZEGORCZYK: “On the Definitions of Computable Real Continuous Functions”, pp.61–77 in *Fundamenta Mathematicae* **44** (1957).
- [HaHa83] S.W. HAWKING, J.B. HARTLE: “The Wave Function of the Universe”, pp.2960–2975 in *Physical Review D* vol.**28** (1983).
- [Hage82] N. HAGER: “Modelle in der Physik”, vol.**278** in *Wissenschaftliche Taschenbücher*, Akademie-Verlag Berlin (1982).
- [Holl96] P.R. HOLLAND: “Is Quantum Mechanics Universal?”, pp.99–110 in (J.T. Cushing, A. Fine, S. Goldstein, Eds) *Bohmian Mechanics and Quantum Theory: An Appraisal*, Kluwer (1996).
- [Kie03a] T. KIEU: “Quantum Algorithm for Hilbert’s Tenth Problem”, pp.1461–1478 in *International Journal of Theoretical Physics* **42**, Springer (2003).
- [Kie03b] T.D. KIEU: “Computing the non-computable”, pp.51–71 in *Contemporary Physics* vol.**44**(1) (2003).
- [Klee52] S. KLEENE: “*Introduction to Metamathematics*”, Van Nostrand (1952).
- [Koi91] K.-I. KO: “*Computational Complexity of Real Functions*”, Birkhäuser (1991).
- [Kuhn62] T.S. KUHN: “*The Structure of Scientific Revolutions*”, University of Chicago Press (1962).
- [Kush84] B.A. KUSHNER: “Lectures on Constructive Mathematical Analysis”, vol.**60** in *Translations of Mathematical Monographs*, AMS (1984); translated from Russian (Moscow 1973) and edited by E. Mendelson and L.J. Leifman.
- [Lloy02] S. LLOYD: “Ultimate Physical Limits to Computation”, pp.1047–1054 in *Nature* vol.**406** (Aug.2000).
- [Lloy06] S. LLOYD: “*Programming the Universe*”, Random House (2006).
- [LoCo08] B. LOFF, J.F. COSTA: “Five Views of Hypercomputation”, to appear in *International Journal of Unconventional Computing*.
- [Loff07] B. LOFF: “*Physics, Computation and Definability*”, Dissertation, Mathematics Department of the Instituto Superior Técnico, Lisbon, Portugal (Nov.2007).
- [Ludw85] G. LUDWIG: “*An Axiomatic Basis for Quantum Mechanics*”, Springer (1985).
- [Ludw90] G. LUDWIG: “*Die Grundstrukturen einer physikalischen Theorie*”, Springer (1990).
- [MaHa94] B.S. MAJEWSKI, G. HAVAS: “The Complexity of Greatest Common Divisor Computations”, pp.184–193 in *Proc. 1st International Symposium on Algorithmic Number Theory*, Springer LNCS vol.**877** (1994).
- [Matz92] D. MATZKE (Chairman): “*IEEE Proc. Workshop on Physics and Computation*” (Texas 1992), <http://ieeexplore.ieee.org/xpl/RecentCon.jsp?punumber=4863>; see also <http://ieeexplore.ieee.org/xpl/RecentCon.jsp?punumber=2948> (1994).
- [MeZi06] K. MEER, M. ZIEGLER: “Uncomputability Below the Real Halting Problem”, pp.368–377 in *Proc. 2nd Conference on Computability in Europe (CiE’06)*, Springer LNCS vol.**3988**.
- [Miln97] J. MILNOR: “*Topology from the Differentiable Viewpoint*”, Princeton University Press (1997).
- [Mitt72] P. MITTELSTAEDT: “*Die Sprache der Physik*”, B.I. Wissenschaftsverlag (1972).
- [Moor90] C. MOORE: “Unpredictability and undecidability in dynamical systems”, pp.2354–2357 in *Phys. Rev. Lett.* vol.**64** (1990).
- [Myrv95] W.C. MYRVOLD: “Computability in Quantum Mechanics”, pp.33–46 in [DSKS95].
- [Papa94] C.H. PAPADIMITRIOU: “*Computational Complexity*”, Addison-Wesley (1994).
- [PERi89] M.B. POUR-EL, J.I. RICHARDS: “*Computability in Analysis and Physics*”, Springer (1989).
- [PIM06] A. PERCUS, G. ISTRATE, C. MOORE: “*Computational Complexity and Statistical Physics*”, Oxford University Press (2006).
- [Ord02] T. ORD: “Hypercomputation: computing more than the Turing machine”, Honour’s Thesis, University of Melbourne (2002); <http://arxiv.org/math.LO/0209332>
- [Pont07] J. PONTIN: “A Giant Leap Forward in Computing? Maybe Not”, *The New York Times*, April 8 (2007).
- [ReTa93] J. REIF, S.R. TATE: “The Complexity of N-Body Simulations”, pp.162–176 in *Proc. 20th International Conference on Automata, Languages, and Programming (ICALP’93)*, Springer LNCS vol.**700**.
- [RTY94] J.H. REIF, J.D. TYGAR, A. YOSHIDA: “Computability and Complexity of Ray Tracing”, pp.265–276 in *Discrete and Computational Geometry* vol.**11** (1994).
- [Sche97] E. SCHEIBE: “*Die Reduktion physikalischer Theorien*”, Springer (1997).
- [Schm08] H.-J. SCHMIDT: “Structuralism in Physics”, in *Stanford Encyclopedia of Philosophy*, <http://plato.stanford.edu/entries/physics-structuralism/> (2008).
- [Schr96] J. SCHRÖTER: “*Zur Meta-Theorie der Physik*”, de Gruyter (1996).
- [Ship93] J. SHIPMAN: “Aspects of Computability in Physics”, pp.299–314 in *Workshop on Physics and omputation* (IEEE Computer Society, Los Alamitos, NM, 1993).
- [Smi06a] W.D. SMITH: “Church’s Thesis meets the N-Body Problem”, pp.154–183 in *Applied Mathematics and Computation* vol.**178** (2006).
- [Smi06b] W.D. SMITH: “Three counterexamples refuting Kieu’s plan for «quantum adiabatic hypercomputation» and some uncomputable quantum mechanical tasks”, pp.184–193 in *J. Applied Mathematics and Computation* vol.**178** (2006).

- [Snee71] J.D. SNEED: “*The Logical Structure of Mathematical Physics*”, Reidel Dordrecht (1971).
- [Steg79] W. STEGMÜLLER: “*The Structuralist View of Theories*”, Springer (1979).
- [Steg86] W. STEGMÜLLER: “Theorie und Erfahrung”, vol.**II** part.F in *Probleme und Resultate der Wissenschaftstheorie und Analytischen Philosophie*, Springer Studienausgabe (1986).
- [Stoe95] M. STÖLZNER: “Levels of Physical Theories”, pp.47–64 in [DSKS95].
- [StSt71] A. STRUGATSKY, B. STRUGATSKY: “*Roadside Picnic*” (1971).
- [Svoz93] K. SVOZIL: “*Randomness and Undecidability in Physics*”, World Scientific (1993).
- [Svoz95] K. SVOZIL: “A Constructivist Manifesto for the Physical Sciences – Constructive Re-Interpretation of Physical Undecidability”, pp.65–88 in [DSKS95].
- [Svoz98] K. SVOZIL: “*Quantum Logic*”, Springer (1998).
- [Svoz05] K. SVOZIL: “Computational universes”, pp.845–859 in *Chaos, Solitons and Fractals* vol.**25:4** (2005).
- [Yao03] A.C.-C. YAO: “Classical Physics and the Church-Turing Thesis”, pp.100–105 in *Journal of the Association for Computing Machinery* vol.**50(1)** (2003).
- [Wein94] S. WEINBERG: “*Dreams of a Final Theory: The Scientist’s Search for the Ultimate Laws of Nature*”, Vintage (1994).
- [Weiz85] C.F. VON WEIZSÄCKER: “Das Gefüge der Theorien”, Kapitel 6 in *Aufbau der Physik*, Carl Hanser (1985).
- [WeZh99] K. WEIHRAUCH, N. ZHONG: “The Wave Propagator Is Turing Computable”, pp.697–707 in *Proc. 26th International Colloquium on Automata, Languages and Programming (ICALP 1999)*, Springer LNCS vol.**1644**.
- [WeZh02] K. WEIHRAUCH, N. ZHONG: “Is wave propagation computable or can wave computers beat the turing machine?”, pp.312–332 in *Proc. London Mathematical Society* vol.**85:2** (2002).
- [WeZh06] K. WEIHRAUCH, N. ZHONG: “Computing Schrödinger propagators on Type-2 Turing machines”, pp.918–935 in *Journal of Complexity* vol.**22:6** (2006).
- [Wolf85] S. WOLFRAM: “Undecidability and Intractability in Theoretical Physics”, pp.735–738 in *Physical Review Letters* vol.**54:8** (1985).
- [Xia92] Z. XIA: “The Existence of Noncollision Singularities in the N /Body Problem”, pp.411–468 in *Ann. Math.* vol.**135:3** (1992).
- [Zieg03] M. ZIEGLER: “Computational Power of Infinite Quantum Parallelism”, pp.2057–2071 in *International Journal of Theoretical Physics* vol.**44:11** (2005).
- [Zieg06] M. ZIEGLER: “Effectively Open Real Functions”, pp.827–849 in *Journal of Complexity* vol.**22** (2006).
- [Zieg08] M. ZIEGLER: “(A Meta-Theory of) Physics and Computation”, p.145 in *Verhandlungen der Deutschen Physikalischen Gesellschaft (DPG)*, Feb.2008.
- [Zuse06] “Is the Universe a Computer? From Konrad Zuse’s Invention of the Computer to his «Calculating Space» to Quantum Computing”, Symposium under the patronage of Dr. Annette Schavan, Federal Minister of Education and Research (Nov. 2006). http://www.gi-ev.de/regionalgruppen/berlin/download/veranstaltungen/ist_das_universum_flyer.pdf